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Recombinant Enaction: Manipulatives Generate New Procedures in the Imagination, by Extending and Recombining Action Spaces

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Abstract

Manipulation of physical models such as tangrams and tiles is a popular approach to teaching early mathematics concepts. This pedagogical approach is extended by new computational media, where mathematical entities such as equations and vectors can be virtually manipulated. The cognitive and neural mechanisms supporting such manipulation-based learning—particularly how actions generate new internal structures that support problem-solving—are not understood. We develop a model of the way manipulations generate internal traces embedding actions, and how these action-traces recombine during problem-solving. This model is based on a study of two groups of sixth-grade students solving area problems. Before problem-solving, one group manipulated a tangram, the other group answered a descriptive test. Eye-movement trajectories during problem-solving were different between the groups. A second study showed that this difference required the tangram's geometrical structure, just manipulation was not enough. We propose a theoretical model accounting for these results, and discuss its implications.

Keywords: Manipulatives; Distributed cognition; Enactive cognition; Forward models; Epistemic action; Transfer

1. Introduction

Manipulable instructional aids play a key role in learning-by-doing and constructivist educational approaches in general, particularly in the learning of mathematics at the

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primary and middle school levels (Boggan, Harper, & Whitmire, 2010; Martin, Lukong, & Reaves, 2007; Tchoshanov, 2011). The popularity of manipulatives as teaching tools is supported by studies showing that they scaffold the learning of both arithmetic and geometry (Martin & Schwartz, 2005; Olkun, 2003; Tchoshanov, 2011; Uttal, Scudder, & DeLoache, 1997). However, the interactive process by which manipulatives change the cognitive system—particularly the way actions executed on these external structures lead to the generation of internal structures—is not well understood. As manipulation-based learning of mathematics is popular and successful, this practice provides a well-structured paradigm instance to develop a theoretical model of the way knowledge emerges through interaction with external structures. A central objective of this paper is developing such a theoretical model.

Apart from the theoretical interest in understanding the way interaction and manipulation generates internal structures, there is also significant application interest in this problem, as the design of new computational media for mathematics and science learning is mostly based on multi-touch and embodied manipulation of formal entities (Sarama & Clements, 2009b). Recent applications include systems to learn numbers (Sinclair & De Freitas, 2014), algebra (Ottmar, Weitnauer, Landy, & Goldstone, 2015; Weitnauer, Landy, & Ottmar, 2016), vectors (Karnam, Agrawal, Mishra, & Chandrasekharan, 2016), proportions (Shayan, Abrahamson, Bakker, Duijzer, & Van der Schaaf, 2015), volume (Lakshmi, Narayana, Prasad, Murthy, & Chandrasekharan, 2016), and equations and graphs (Majumdar et al., 2014). The design of such embodied interaction systems for learning are of interest from a theoretical perspective as well (Abrahamson & Sánchez-García, 2016; Hutto, Kirchhoff, & Abrahamson, 2015), because such designs work in dual mode—as educational interventions as well as probes into the cognitive system—thus providing insights into the way embodied interactions lead to the generation of internal structures. A related approach examines the role of gestures (Alibali & Nathan, 2012; Goldin-Meadow, Cook, & Mitchell, 2009; Novack, Congdon, Hemani-Lopez, & Goldin-Meadow, 2014) in mathematics learning. Actions on new computational media are also considered similar to the process of gesturing and drawing during the mathematical discovery process (De Freitas & Sinclair, 2014). These overt movements are hypothesized to be part of the mechanism that helps move body-based intuitions (about possible mathematical results) into externalized symbolic proofs, which are built using known and accepted mathematical structures (also see Marghetis & Núnez, 2013; Rotman, 2008; Sfard, 1994).

Many models have been proposed to account for the way manipulatives contribute to the learning of early mathematical concepts such as fractions and area. One approach considers manipulatives as a specific instance of multiple representations, which help provide different perspectives (visual, symbolic etc.) of the same concept, and thus improve students' understanding of the underlying mathematical principles (Moreno & Mayer, 1999). A second approach is the cognitive off-loading hypothesis, which suggests that physical manipulatives help distribute working memory load to the external environment, and this allows students to perform more sophisticated mental calculations than what their internal memory resources alone would support (Cary & Carlson, 1999). Finally, an action priming account suggests that manipulatives work by analogy, specifically analogous actions: “the best manipulatives are the ones that require physical manipulations that are analogous to the abstract mental manipulations required by the problem” (Hall, 1998). This analogous action account does not rule out the first two accounts (multiple perspectives, working memory offloading), but it focuses more on the connection between procedures, which are embedded in the manipulative and the target concept to be learned.

In contrast to these descriptive approaches, we are interested in understanding the *mechanisms* by which manipulatives support the learning of mathematics and science concepts, and seek to develop a *mechanism model* of this interaction process. This is a cognition problem, which we study using a task from education (calculating area). In this paper, we seek to address the following questions:

1 What *cognitive mechanisms* could account for actions in the manipulatives turning into thinking?
2 What *neural mechanisms* could support this transformation, and how?

To address these two questions, we develop a new process-oriented study method, combining qualitative approaches (from education and problem-solving research) with an eye tracking analysis based on transition matrices (an analysis method from neuroscience). The main components of this approach are not new, as both qualitative studies and eye tracking have been used for more than 40 years to study children’s problem-solving, starting with studies of “centered” and “decentered” eye fixations during the volume conservation task (O’Bryan & Boersma, 1971). We just extend these approaches, to develop a novel analysis method that seeks to characterize task-oriented eye movements, where the eye is systematically moved to different locations in the task space during problem-solving. In this approach, the eye is treated as an actuator, and its movements are analyzed similar to the way the task-oriented movements of the hand are tracked during problem solving tasks such as Tower of Hanoi. Eye fixations during such problem-solving are considered to mark shifts in executive attention (Smith & Kosslyn, 2007), which allow the problem solver to track and integrate micro-level moves, sequences, and task switching during the solving of the problem.

Data from two studies based on this method and analysis (Section 2) provide insight into the way manipulatives change the problem-solving process. We treat the results from this study as indicative, that is, not providing a conclusive empirical case. However, given the rather nuanced structure of the study results, we find them useful to develop models of the cognitive and neural mechanisms involved. To address Question 1 above, we outline a way actions done on manipulatives could turn into internal procedures (Section 3). Addressing Question 2, we develop a neural mechanism model, based on three neural processes proposed by recent work in cognitive and motor neuroscience—common coding (of execution, perception and imagination of action), forward models, and the extension of body schema through tool use. We discuss how these neural-level processes could support the cognitive level model (Section 4). We then examine the theoretical implications of these two mechanism accounts (Section 5), focusing on the following three broad areas of cognition.

1.1. Distribution of cognition

External actions and structures generated during problem solving have been shown to lower cognitive load and improve task performance (Hutchins, 1995; Kirsh, 2010; Kirsh & Maglio, 1994). Such distribution of cognition provides clear cognitive advantages (Kirsh, 2010). However, it is not clear which cognitive and neural mechanisms support the process of distributing cognition. Also, traditional distributed cognition (DC) accounts describe the way internal processing, particularly cognitive load, is offloaded to external actions and artifacts. The other direction, where internal cognitive structures are generated from external actions and structures, is not well studied.

The practice of using manipulatives to help students learn formal concepts at the school level provides a very well-structured problem to study the external-to-internal shift, and the action-embedded internal structures that are generated by this process. Understanding the mechanisms underlying such "physically distributed learning" (Martin & Schwartz, 2005) could thus provide insight into the way internal structures, mechanisms and processes emerge from external actions. Since the internal structures generated by this interaction process embed action, they could trigger related actions during other complex tasks. Such action-embedded internal structures could thus support *the process of distributing cognition*, through the generation of external actions and artifacts during problem solving.

To illustrate the possibilities offered by this approach to the study of mechanisms underlying DC, we propose a mechanism account of the emergence of epistemic actions.

1.2. The transfer process

Transfer of learned concepts and procedures to a new domain is a central objective of education. Most current models of transfer provide structural accounts based on analogy, focusing on the systematic relations between the source and target domains. The process by which transfer occurs is less discussed (Nersessian & Chandrasekharan, 2009), and the mechanism basis of transfer is currently unclear. A mechanism understanding of manipulative-based learning, particularly learning of early mathematics concepts, could provide insights into the process by which transfer occurs, as manipulative-based learning could be considered an instance of transfer. We discuss how the mechanism model we propose could be extended to understand the transfer process.

1.3. Embodied cognition

Recent theoretical work seeks to develop an understanding of the way actions and sensorimotor processes are re-adapted to learn complex concepts (Landy, Allen, & Zednik, 2014), and higher order cognition in general (Gallese & Lakoff, 2005). A significant focus of this work is the way language is processed using simulative mechanisms (Bergen & Wheeler, 2010; Matlock, 2004; Yee, Chrysikou, Hoffman, & Thompson-Schill, 2013). Manipulatives provide a way of extending this embodied cognition approach to other domains, particularly science and mathematics, adding to work already under way in this direction (Chandrasekharan, 2009, 2013, 2014; De Freitas & Sinclair, 2014; Goldin-Meadow & Beilock, 2010; Nersessian, 2010). The model we develop here illustrates an instance of such an extension, based on mechanisms proposed in cognitive and motor neuroscience.

One central issue that needs to be addressed in extending embodied cognition to formal knowledge domains is the way embodiment is related to stored symbols and imagination processes, as these play significant roles in formal problem-solving, as well as during learning and use of mathematics and science concepts. It is currently unclear how these internal aspects of formal problem-solving and learning could be reconciled with embodied cognition, particularly approaches such as dynamic systems theory (Smith & Thelen, 2003; Thelen & Smith, 1996; Van Gelder, 1998) and ecological psychology (Gibson, 1979; Reed, 1996), which have been *interpreted* as rejecting internal representations and processes based on them (Chemero, 2000). We argue that a commitment to embodied cognition and dynamic systems approaches does not automatically require the rejection of internal representations and imagination processes, as these can be understood as dynamic control mechanisms that emerge through interaction with the environment (Chandrasekharan & Osbeck, 2010; Chandrasekharan & Stewart, 2007; Van Gelder, 1998). Such emergence of internal representations and processes could be seen as a form of state change (such as gas to liquid, liquid to solid, etc.), a shift compatible with views emphasizing dynamics and control.

The next section discusses an illustrative study of sixth grade students solving area problems. The results from this study ground the mechanism model we propose. The study seeks to develop just an account of how working with manipulatives changes the problem-solving process. It does not provide a longitudinal account of the way a formal concept (such as area or fractions) is learned, gradually through interaction with manipulatives. This is a more complex problem. However, the mechanism account of manipulative-based problem-solving we propose here could play a supportive role in developing models of this extended learning process.

2. Empirical study: How manipulation changes the process of solving area problems

The use of manipulatives in learning, while popular with teachers and supported by results from individual research studies, is not fully supported by converging evidence, as meta-analyses and reviews, of studies comparing groups and classrooms that use and do not use manipulatives, show no overall advantage for manipulative use (McNeil & Jarvin, 2007; Sowell, 1989; also see Bosse, 2016). One way to start the process of reconciling this divergence between classroom practice and overall research findings is to characterize in detail:

1. The specific cognitive changes, if any, induced by manipulatives in the *cognitive process* of solving particular mathematics problems.

2. The ways in which these specific shifts in the *cognitive process* contribute to the understanding of mathematics concepts.

Based on such studies of the cognitive process, the specific contributions of manipulatives, in helping the student understand/generate mathematics procedures and concepts, could be characterized. Educational interventions could then be designed to exploit these specific cognitive elements. This type of cognitive analysis, seeking to characterize in detail the micro-level interactions involved in the problem-solving process, is the broad research approach we follow. This type of process accounts is the focus of much of education research, but results from these studies are usually not used to develop mechanism accounts. The specific education domain examined in the analysis here is the problem of calculating area. We characterize how the cognitive process involved in solving area problems changes after working with manipulatives. This process account is developed through the tracking of hand and eye movements.

In the studies reported here, the eye tracker is used as a micro-level observation device (similar to a microscope), to generate a highly detailed qualitative picture of the task-oriented eye movements during the problem-solving process. To develop an understanding of how the task-oriented eye movements change with the problem-solving context, we used a two condition (baseline, study) intervention approach. This study approach is similar to qualitative field studies in ethology and anthropology, as well as classroom studies in education, where controlled interventions are often used in combination with qualitative observation, to generate and characterize problem-solving behavior. The characterization involved is similar to qualitative studies of problem-solving, which tracks the task-oriented actions and moves in the problem-space. Extending this method to eye tracking, we focus on task-oriented eye movements, and thus the actuator role of the eye, rather than the perceptor role. In the analysis approach we use, the fixation data are treated as an indicator of shifts in executive attention, which helps control and track the task-oriented movements of the eye (visits, returns, sequences, switching etc.) while solving the problem.

We studied the problem-solving process related to the area concept, which is a key transition point in the learning of mathematics, as learning area requires bringing together, in an integrated fashion, spatial, numerical, and some rudimentary algebraic ideas, as well as shifting between them. Area is thus one of the entry points to the deep interconnections between these three components of mathematics (Rahaman, Subramaniam, & Chandrasekharan, 2013; Sarama & Clements, 2009a), and students find learning the area concept very difficult (Battista, 2007). To begin with, learning area as an array of units requires “breaking” a given figure into units (partitioning), counting the units, and understanding the notion of conservation—that is, the units rearranged in any spatial pattern would have the same area. At the next level of complexity, area requires understanding fractions and multiplication, as both these concepts are involved in calculating area symbolically (Outhred and Mitchelmore, 2000). In the other direction, these concepts are also strengthened by an in-depth understanding of the area concept (Ball, Lubienski, & Mewborn, 2001). At the third level, a rudimentary notion of algebra is required to calculate area, say of a rectangle using the *Length* × *Breadth* ($L \times B$) formula. Finally, the mathematical abstraction of limit, which is embedded within the area concept, needs to be understood, as the shift to using measurements of one-dimensional sides, instead of counting of two-dimensional units, requires imagining the units getting smaller and smaller until they become points in a line.

All these elements together *constitute* (Landy et al., 2014; Pande & Chandrasekharan, 2017; Sfard, 2000) the area concept. Area can thus be thought of as a network concept that integrates all these elemental concepts. The area network is very rich and complex (see Fig. 1), and establishing the interconnections between the disparate concepts involved is a central difficulty in learning area. Our pilot studies show that the area concept, and its connection to other concepts, tend to be patchy and unstable (Rahaman et al., 2013). We also found that students tended to follow algorithmic approaches, such as $L \times B$, and did not understand the spatial relation between unit and area. Workshop and interview sessions with students also showed that the limit concept, involved in moving from counting units to the $L \times B$ abstraction, is very difficult to grasp. The students also found the related distinction between perimeter and area difficult. These constructs (spatial connection between unit and area, unit-measurement relation, the perimeter/area distinction, and the integration of conceptual elements) are the key features that need to be strengthened, in an integrated fashion, by interventions that seek to teach area.

Manipulation of physical dissection models, such as assembling of unit-based figures, tiling and covering using units, and so on, are standard teaching approaches to make the area concept easier to learn (Outhred & Mitchelmore, 2000). Intuitively, this appears to be an effective (and required) approach to teach area, as area is a property of physical entities. However, it is not clear how these physical manipulations help in understanding the formal notion of area, particularly in bringing together the different mathematical constructs involved in the area concept. To understand this process, the studies we report here use an intervention similar to the use of manipulatives in classrooms, to examine how working with manipulatives just before doing two area tasks changes the process of solving the area problems. The results suggest that students who worked with the manipulative intervention chunk the test figure in different ways, and calculate area using a real-time approach to change the composition of the given figure using smaller figures. However, this strategy does not lead to significant improvement in accuracy. This nuanced set of results demonstrate how learning by doing could be seen as failing (as indicated by the meta-analysis result discussed above), even when the manipulative intervention changes the problem-solving *process* in the right direction—a case of partial transfer (Bransford & Schwartz, 1999). This dissociation between process and final solution could partially account for the conflict between teaching intuitions/practice and the meta-analysis result.

To systematically develop designs that address this dissociation, particularly new media designs, a general cognitive account of how interaction with manipulatives changes the process of solving problems is needed. Here, we develop such a general cognitive account, outlining possible mechanisms that underlie the changes in problem-solving process generated by manipulatives. The primary contribution of the paper is thus theoretical, particularly the development of an enactive cognition approach to understand the way manipulatives change the problem-solving process.

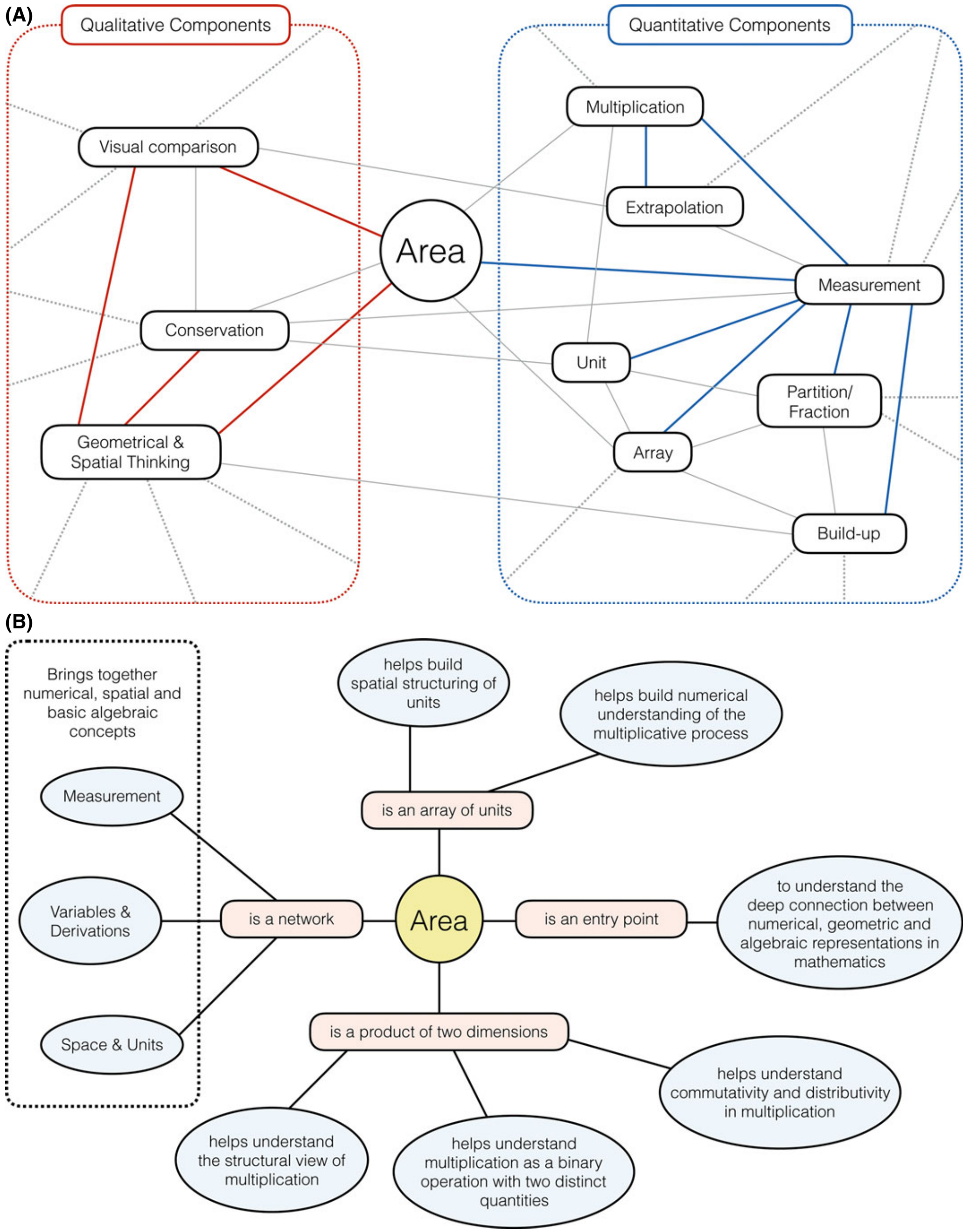

Fig. 1. (A) Graphic illustrating how the different components of the mathematical concept of area could be understood as a network (top), and (B) a concept map of how area is related to other concepts (bottom). “Build-up” refers to the ability to generate a figure of a given area using smaller units. “Extrapolation” refers to the ability to apply the area notion to measure a large rectangular space (such as a tabletop or a room), either using a square unit and unit of units, or using measurement and the $L \times B$ formula.

We would like to emphasize here that this paper only draws inspiration and guidance from the pedagogical use of manipulatives in solving mathematical problems, particularly area. Our objective here is developing a theoretical model of the cognitive mechanisms underlying manipulative-based learning. So the focus is on how manipulatives change the problem-solving process, and which cognitive and neural mechanisms support this change. We do not seek to provide an account of the way manipulatives (over a period of time) help solve the complex integration problem involved in learning area. Understanding this integration problem requires a wider set of studies, which could use the model we propose here as a starting point.

2.1. Study 1

The first study examined the following question: *What change in the cognitive process, if any, is induced by a physical manipulation task when students are trying to solve area problems?* To answer this question, two area calculation tasks were given to participants. Before starting the two area tasks, participants completed one of two pre-tasks—either manipulating and solving a single tangram-like puzzle (study group), or answering general knowledge questions (baseline group). Participants were randomly assigned to this pre-task condition.

Tangram is an old Chinese puzzle, where seven geometrical pieces are manipulated to generate various figures. Our manipulation task only had four pieces, as participants in the pilot testing phase found the seven-piece tangram too difficult to solve. Olkun (2003) reports that experience with solving tangrams, both concrete and computer-based, has a positive effect on students' two-dimensional geometric reasoning. Tangrams can also play an important role in the development of spatial ability, competency of rotation and space, geometrical knowledge, reasoning, geometrical imagination, and conservation of area (Baran, Dogusoy, & Cagiltay, 2007; Chiu-Pin, Shao, Lung-Hsiang, Yin-Jen, & Niramitranon, 2011). Tangrams can also be used to bring together different domains of mathematics, such as number sense, algebra, geometry, and measurement (Tchoshanov, 2011).

2.1.1. Participants

Twenty-two grade 6 students (11–13 years; 12 female, 10 male) from two Mumbai schools (11 from an English medium school, 11 from a Marathi medium school) participated in the study. Participants were assigned to baseline and study groups randomly, giving us 11 participants in each group. The students and their parents provided consent before participation.

Area is introduced in grade 5 in India. The students thus had 1 year formal exposure to area as a mathematical concept. The teaching focus in schools is on learning the $L \times B$ algorithm, based on example pictures in the textbook, which are also drawn on the blackboard by the teacher. Manipulatives are not used in the classrooms, particularly in the schools we studied, which are in a middle- to low-income neighborhood.

2.1.2. Task

The primary task in our study was to calculate the area of two non-standard figures (see Fig. 2) drawn on graph paper. The second figure (B) was given after the first one (A) was solved. A unit was shown on one corner of the graph paper, and students were asked the following area-problem question: *A full cake is shown in the figure. A piece of this cake is shown at the right corner of the graph paper. This piece costs Rupees 1/-. What will be the cost of the entire cake?* The question was rephrased or translated if needed. No time limit was set for completing the task. Most students completed the task and the following interview in 45 minutes.

Each student performed the task individually on a table, sitting in a height-adjustable chair. Before starting the area task, the study group was required to make a square out of four cardboard pieces of different shapes (see Fig. 2 for the precise shapes). The baseline group was asked some general knowledge questions before they started the area task. The general knowledge questions were given so that both groups got a pre-task, and both completed them successfully. The students were settled in by a friendly researcher, who emphasized that all the tasks were exploratory and did not involve any kind of assessment.

2.1.3. Data sources

2.1.3.1. Eye movements: A video camera (Logitech C 525, HD 720p Autofocus) was aligned vertically above the work surface. The video from this camera was synchronized

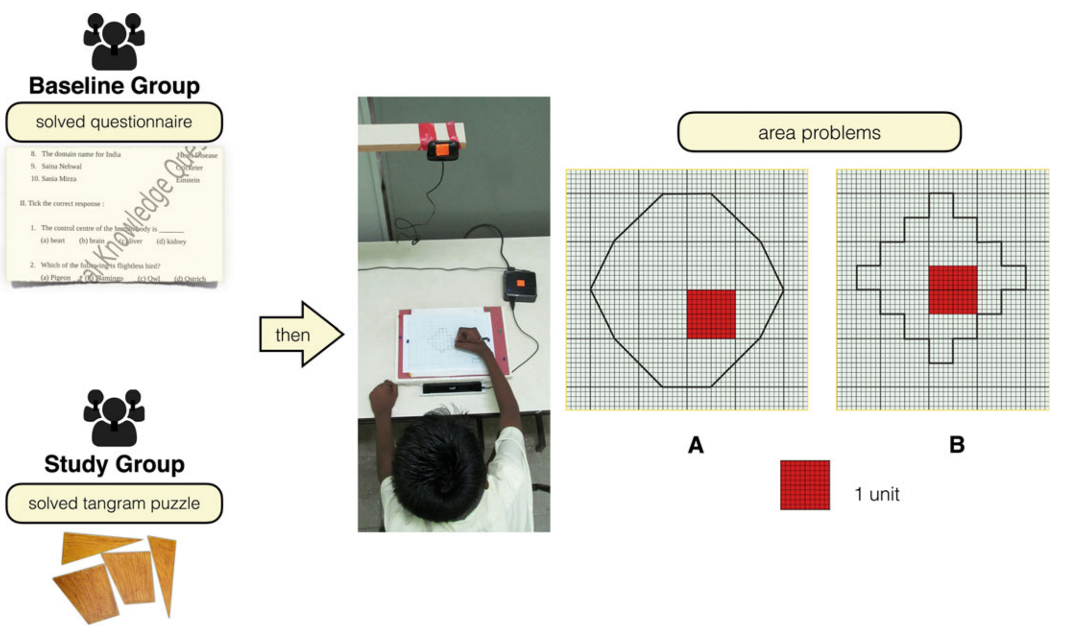

Fig. 2. Graphic illustrating the design and the physical elements of our experiment. All participants calculated the area of the two figures drawn on graph paper shown to the right. Area calculations were restricted to be performed within the working surface of our customized work-bench.

with a Tobii static eye-tracker (Tobii Technology, Stockholm, Sweden), which was mounted on the work surface (see picture in Fig. 2 for positioning). This non-standard configuration of the static eye tracker (which is usually used to track eye movements on a laptop screen) was developed in collaboration with Tobii technical personnel, who provided onsite help to calibrate the system. This setup allowed tracking of participants' task-oriented eye movements on the graph paper as they worked on the area calculation tasks. This setup was needed because we could not do the task on a computer, for three reasons. One, we needed to track pencil movements and marks on the graph paper. Second, many of the students we were working with came from a low-income neighborhood and were not familiar with computers. Finally, the task is more intuitive on paper than on a computer.

2.1.3.2. Video: A separate camera was set up on a tripod in a corner of the room, and it captured video data for the pre-task, the area tasks, and the post-task interviews.

2.1.3.3. Pencil marks and movements: Participants were given a pencil and told that they could make markings on the graph paper. They were instructed to write the final answer for each problem on the graph paper sheet.

2.1.3.4. Interviews: Each participant was interviewed post-test, and was asked how they approached the problem and how they solved it. Some students changed their answer to the cake problem during the interview. They were asked about the strategy they used during the task and why they changed their answer.

These data sources gave us multiple windows into the problem-solving process. Apart from these process data, final answer values and time taken for the solutions were also collected for each participant.

The primary focus of our analysis was eye movements. The accuracy data and hand movement process data were analyzed first to develop a qualitative approach to characterize the task-oriented movements of the eye during problem solving.

2.1.4. Data analysis

Here we outline the methods used for analysing the video data and eye tracking data.

2.1.4.1. Video data: To tease out possible differences in the strategies used by students, we did a qualitative analysis of the video recordings and participant interviews. The different calculation strategies used by participants were coded systematically, in two phases: one, using their pencil movements, paper-based markings and calculations recorded on video, and second, the strategies they reported in the post-task interviews. The coding scheme for videos was developed using two students' videos and interviews. This scheme, after discussions in the research group, was fixed for all the other videos. These codes were then validated against self-reported strategies in post-task interviews. Table 1 shows the coding scheme that was used for the qualitative analysis of the video data.

2.1.4.2. Eye movement data: Based on the above qualitative analysis, we moved to a characterization of the patterns of task-oriented eye movements corresponding to the spatial chunking and counting strategies (indicated by the qualitative analysis of videos and interviews). The qualitative analysis suggested that the tangram group made large shifts within the task figures, and some participants used a style of partitioning that combined elements in different ways (see results of the qualitative analysis in the next section). This implied that they approached the area task in a global, whole diagram fashion, dividing the whole diagram up into manageable, possibly non-contiguous components, and adding each of these components separately.

In terms of task-oriented eye movements, this approach would be indicated by:

1. the eye movement pattern staying stable within the subtasks (components), and
2. when the eye movement pattern changes, it changing to a greater extent in physical space.

On the other hand, for participants who use a primarily numerical strategy, just counting squares locally, the task-oriented eye movement patterns will change more often, but also more incrementally. It was these two specific patterns that we sought to identify in the eye tracking data.

Note that the area problems cannot be solved by depending on just the numerical strategy (partial elements need to be combined once the countable squares are done), so the two task-oriented eye movement strategies would be used by everyone, but possibly at different levels. Our analysis sought to identify whether systematic patterns existed in the use of these strategies, relative to the intervention conditions.

One standard approach toward characterizing eye movement data is based on an assumed bijective relation between fixations and visual attention. The focus here is on the role of the

Table 1
Strategies Used by Students

Strategy Name	Description
1. Pointing	Just pointing at a unit or part, without marks
2. Marking	Making marks on unit or part
3. Numbering	Putting numbers (1, 2, 3) to units or parts, or adding them using numbers
4. Closer shifting	Shifting parts to immediate neighbors or adjacent positions
5. Distant shifting	Shifting parts to places other than immediate neighborhood, and also moving parts from within units
6. Making outlines (Explicit partitioning)	Making marks or outline for parts/partitions
7. Assigning numerical values to parts	Explicit mention of the values for the parts, giving fractional or decimal values to parts
8. Estimating	Estimating values of different units. Could be any 4 quarters making 1 unit or any two halves making 1 unit
9. Approximating (not numerical or geometrical)	Values where it is not clear how a particular value is assigned by the student. Also values reported by students without justification for why that value was assigned
10. Totaling	Final adding together of units or parts

eye as a perceptor, where the information distribution in the given figures guides the eye to fixate on specific areas. In this analysis, fixations and saccades track visual attention and the way it shifts, and patterns in this data provide an indication of the way the eye gathers information. Following this approach, the pattern of behavior we are looking for could, intuitively, correspond to the expectation that the experiment group would show:

(P) Longer fixations on average, since stable gaze within subtasks is likely to be construed as long fixations by velocity-sensitive fixation classification algorithms, and
(Q) Greater kurtosis in the distribution of saccade lengths, since the saccades within and across subtask components would be heterogeneous in size, reflecting the hierarchical nature of the eye's engagement with the task.

Fig. 3 outlines the results of this analysis: While participants in the experiment group did seem to have longer fixations on average, the difference is not statistically significant ($.2 < p < .25$ in both tasks). Whereas we expected the kurtosis of the experiment group's saccade length distribution to be larger, these were approximately equal in both samples for both tasks (Task A: {5.26, 5.07}, Task B: {6.06, 5.03} for the control and tangram groups, respectively).

The inconclusive nature of this analysis stems considerably from its generality. By treating fixations and saccades as the basic unit of analysis, we ignored the task-relevant

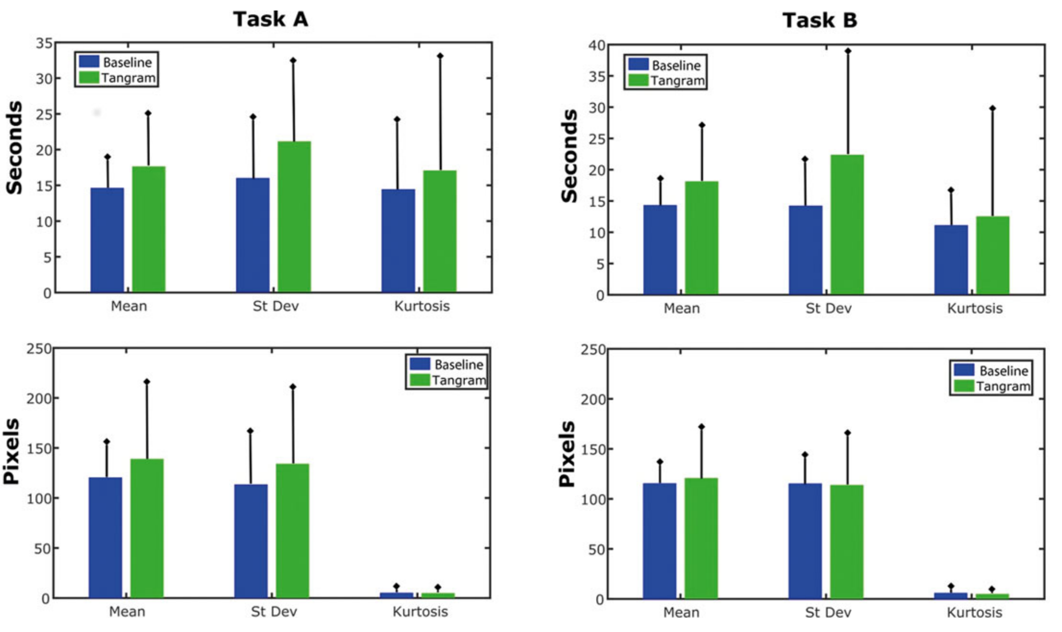

Fig. 3. Fixation duration and saccade length statistics for eye movement data for the baseline and tangram groups in our experiment. Bars show standard deviations for the specific statistic. The intervention group had longer fixations on average, but the difference was not statistically significant. Kurtosis measurements for saccade length were not significantly different across both groups. The fixation analysis was done using Tobii's velocity-based classification filter, using default parameters.

spatial locations of the actual area subunits that our participants are manipulating internally to solve the problems. Remaining agnostic about the task-relevant spatial contents of the scene is a useful strategy for a general analysis of eye movement patterns, particularly to understand the way information is clustered in figures and texts, and how this clustering directs visual attention and perception. But this approach is suboptimal in tracking the differences in task-oriented actions and moves in a problem-solving space. The task-oriented movements of the eye engages executive attention, which is involved in monitoring, sequencing, and task switching during problem solving (Botvinick, Braver, Barch, Carter, & Cohen, 2001; Fernandez-Duque, Baird, & Posner, 2000; Smith & Kosslyn, 2007). Our analysis sought to identify fixations that embed these executive attention processes, which are closely tied to task-relevant transitions. An analysis based on all fixations and saccades would not focus on these executive attention elements, and thus not allow the characterization of the differences in the task-oriented actions and moves related to problem-solving.

To characterize the task-oriented eye movements in our tasks, and to analyze and identify possible patterns across the two conditions, the following steps were done:

(a) each area image was divided into sets of states
(b) transition probabilities between these states were calculated
(c) task-relevant transitions by frequency were identified within fixed time windows
(d) large changes in the set of relevant transitions between contiguous time windows were identified (as a marker of shift in gaze patterns).

First, we divided each of the area test diagrams into a set of states S and identified their coordinate locations for each eye-tracking study setting. Fixation data for each participant then became a finite string of state occurrences $s \varepsilon S$. Then, we calculated pairwise transition probabilities for each state s_a with respect to all other states, such that

$$p(s_a \to s_b) = \frac{n(a \to b)}{n(a \to)},$$

to obtain transition matrices T, s.t. $T_{ab} = p(a \to b)$ for each participant.

We further obtained a matrix enumerating salient transitions, by selecting transition probabilities that were not too low ($p < 1/(|S| - 1)$), ignoring self-transitions, and transitions between states outside the diagram. Thus, we obtained a binary matrix containing salient transitions per participant M, s.t. $M_{ab} = 1 \Leftrightarrow T_{ab} \in [1/(S \vee -1),]$ and 0 otherwise. While binarizing the transition matrix throws away information about the absolute value of the transition frequencies, these depend heavily on the size of the time window chosen, which is a free parameter in our account. Binarizing the transition matrix provides us information more resilient to the value of this parameter. As a quantitative measure of this resilience, comparing the entries of this binarized matrix obtained using all time window sizes between 1 and 10 s, we obtain a median mismatch rate = 0.053 ± 0.031 averaged across all participants, and all pair-wise state comparisons.

We divided every student's overall gaze sequence up into equal-sized time segments (t = 5 s for all our results below). In this way, we obtained an incremental view of which patterns emerge and which fall away as the participant progressed through the task. Videos showing the evolution of new gaze patterns for each student while completing the first task were made. In each video, faint blue lines marked salient existing gaze patterns and dark blue segments indicated the emergence of a new gaze pattern.

The changes in these videos were too dense and rapid, so it was not possible to make qualitative judgements of the micro-level changes captured by these videos. To gain insight into the micro-level changes, we summarized this information quantitatively, by measuring the extent of change in pattern as the quantitative difference between two transition matrices via the Frobenius norm,

$$\delta_t = \sqrt{\sum_a \sum_b |M_{ab}^{(t+1)} - M_{ab}^{(t)}|^2}.$$

We tested both our postulates using this metric (of the extent of change of the gaze pattern) defined above. Specifically, postulate 1 was operationalized as follows: the average change for the baseline group (no tangram) will be larger than that for the study group (tangram). Postulate 2 was operationalized as follows: the maximal change in the study group will be larger than the maximal change in the baseline group.

2.1.4.3. Study to validate the analysis method: Since the measurement of transition probabilities in this analysis is contingent on the time window size we have used, we also conducted a post hoc validation study to verify that, in its current calibration, this analysis is indeed sensitive to differences in transition patterns when one group uses a counting strategy, and another uses a spatial recombination strategy.

To do this, we randomly assigned a follow-up group of 10 adults (age: 17–37, 6 male, 4 female) to one of two cohorts, both solving the same area tasks as in our main study design. One cohort was instructed to solve the problems using the counting strategy; the other was asked to solve them using the "chunking" strategy. The instructions were as follows: "*Use counting of the units as your dominant strategy to calculate the given area. Trace your actions using the pencil*" (counting strategy), and "*Use shifting of parts to make whole units as your dominant strategy to calculate the given area. Trace your actions using the pencil*" (chunking strategy). Both cohorts were asked to make pencil movements while solving the area problem, so that we could make sure that they were following the instruction.

If our δ measurements are indeed sensitive to difference in strategy, then an observer blind to the identity of the cohort assignments should be able to identify the strategies using the δ values themselves, using the criteria that participants using a chunking strategy should have a lower mean(δ) and a higher max(δ). To see whether this was possible, the eye tracking data from the 10 participants were first anonymized and named using numbers in a random fashion. This dataset was then sent to the research group member then based in the United States. He was blind to the cohort identities but was aware that both cohorts had five participants. He processed the eye-tracking data for all 10 participants and operationalized the criteria above to assign cohort labels to them. For both criteria, we simply summed the scores obtained across both tasks and then ranked them to split our sample into 5/5 cohorts, with the mean criterion assigning the lowest ranks to the chunking cohort and the max criterion doing so for the highest ranks.

The mean criterion correctly predicted 8 out of 10 labels; the max criterion correctly predicted 10 out of 10 labels (Table 2). This performance is clearly better than chance for the max criterion (95% confidence interval for best fit binomial distribution excludes $p = .5$) and almost certainly so for the mean criterion (95% confidence interval for best fit binomial distribution $p = \{.02, .55\}$). Thus, while our measurement of changes in the transition pattern does depend on parameters contingent to the time-scales of our particular study, the results from the validation study suggest that it is well-calibrated to pick out the changes we set out to identify.

2.1.5. Results

2.1.5.1. Area estimation accuracy was not different across the two groups: The first element of our analysis was designing a measure of competence in area calculation. Simply counting how many students got the problem right showed that more students got the right answer in the tangram group for both diagrams, but there is very little difference between the tangram and baseline group (see Fig. 4 for a compilation of our outcome analysis). Since counting the number of correct responses penalizes answers that are quite close to the correct value and wild guesses equally, an alternative measure, the percentage error off the true value, was calculated. Even by this measure though, the error percentages of the two groups were not significantly different. These findings are in line with earlier observations in related work. In a similar study, Olkun (2003) found that the posttest scores for a mathematical problem set were statistically indistinguishable between the test group primed with physical manipulatives and a control group.

Table 2
Validation Study Results

	Mean		Prediction	Max		Prediction	True Label
Subject	Task 1	Task 2		Task 1	Task 2		
1	17.7	13.3	Chunk	27	17	Chunk	Chunk
2	18	17.3	Count	25	13	Count	Count
3	24.3	19.7	Count	23	9	Count	Count
4	14.7	6.3	Chunk	31	20	Chunk	Chunk
5	28.7	10.0	Count	18	18	Count	Count
6	15	15.0	Chunk	39	12	Chunk	Chunk
7	17.3	5.0	Chunk	21	10	Count	Count
8	25	8.0	Count	41	16	Chunk	Chunk
9	25	10.3	Count	25	17	Count	Count
10	15	12.3	Chunk	44	16	Chunk	Chunk

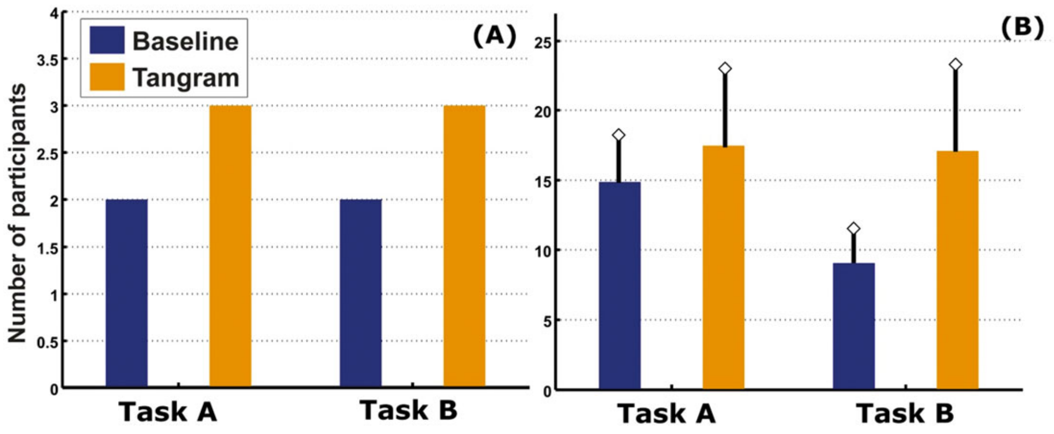

Fig. 4. The tangram intervention does not improve area calculation outcomes. (A) Number of students who solved either area calculation and got the correct answer precisely. (B) Average error percentage in area calculations. Error bar represents 1 *SEM*.

2.1.5.2. Video data and interviews suggest differential strategy use in the tangram group: Both baseline and study populations showed a relatively balanced proportion of strategy use across both tasks, with significant positive correlation in inter-task strategy use within participants (average $\rho = 0.53$). This correlation was computed by binarizing instances of strategy use in both tasks per participant, computing correlations between these two binary vectors, and then averaging correlations across participants. There were, though, potentially interesting deviations in strategy use, particularly in instances of large shifts within the diagram (as measured by finger or pencil movements), explicit partitioning of portions of the diagram with pencil marks, and approximations (coded based only on self-reports). Intuitively, the former two strategies would be expected to be more prevalent in participants who were chunking the space within diagrams and adding the chunks; the latter would be more prevalent in participants who were using simpler counting-based strategies. The general trend of the deviations we observed supports this intuition—the tangram group showed more occurrences of the first two strategies, and considerably fewer of the third. While these deviations were not statistically significant on an individual basis, as is evident from the *p*-values shown in Fig. 5, they provided insight into what patterns to look for in the eye movement analysis.

2.1.5.3. Analysis of task-oriented eye movements revealed patterns consistent with spatial chunking: As discussed in the data analysis section, the qualitative analysis suggested that the tangram group made large shifts within the task figures, and some participants used a style of partitioning that combined elements in different ways. This implied that they approached the area task in a global, whole diagram fashion, dividing the whole diagram up into manageable, possibly non-contiguous components, and adding each of these

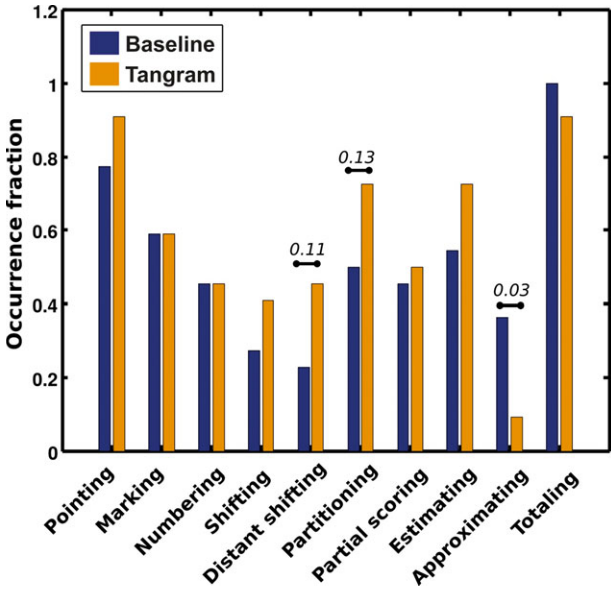

Fig. 5. Qualitative coding of various strategies used by students in the experiment. The behavioral strategies plotted on the *x*-axis were elicited from qualitative coding of task videos and student reports during post-test interviews. Occurrence counted only the use or non-use of a particular strategy by a student, disregarding the number of times they may have used it. Numbers above bar pairs represent *p*-values for corresponding two-sample *t* tests.

components separately. In terms of task-oriented eye movements, this approach would be indicated by the following:

1. the eye movement pattern staying stable within the subtasks (components), and
2. when the eye movement pattern changes, it changing to a greater extent in physical space.

Postulate 1 was operationalized as follows: the average change for the baseline group (no tangram) will be larger than that for the study group (tangram). Postulate 2 was operationalized as follows: the maximal change in the study group will be larger than the maximal change in the baseline group.

Fig. 6 shows the results of the task-specific analysis of eye movements. As can be seen, it is much less noisy than the conventional fixation-saccade analysis provided in Fig. 3. Although postulate 2 is only marginally borne out in task A ($t_{20} = 1.63$, $p = .14$ for difference between baseline and tangram outcomes), postulate 1 is supported strongly ($t_{20} = 2.88$, $p = .0094$) at the $p < .01$ level. For task B, both postulate 1 ($t_{20} = 3.33$, $p = .0033$) and postulate 2 ($t_{20} = 2.95$, $p = .0078$) are strongly supported at the $p < .01$ level. These results together indicate that participants in the tangram cohort made less frequent, but larger jumps in eye gaze patterns, which is the expected gaze signature of the use of the flexible (i.e., many different combinations) spatial partitioning strategies indicated by the qualitative analysis of video data.

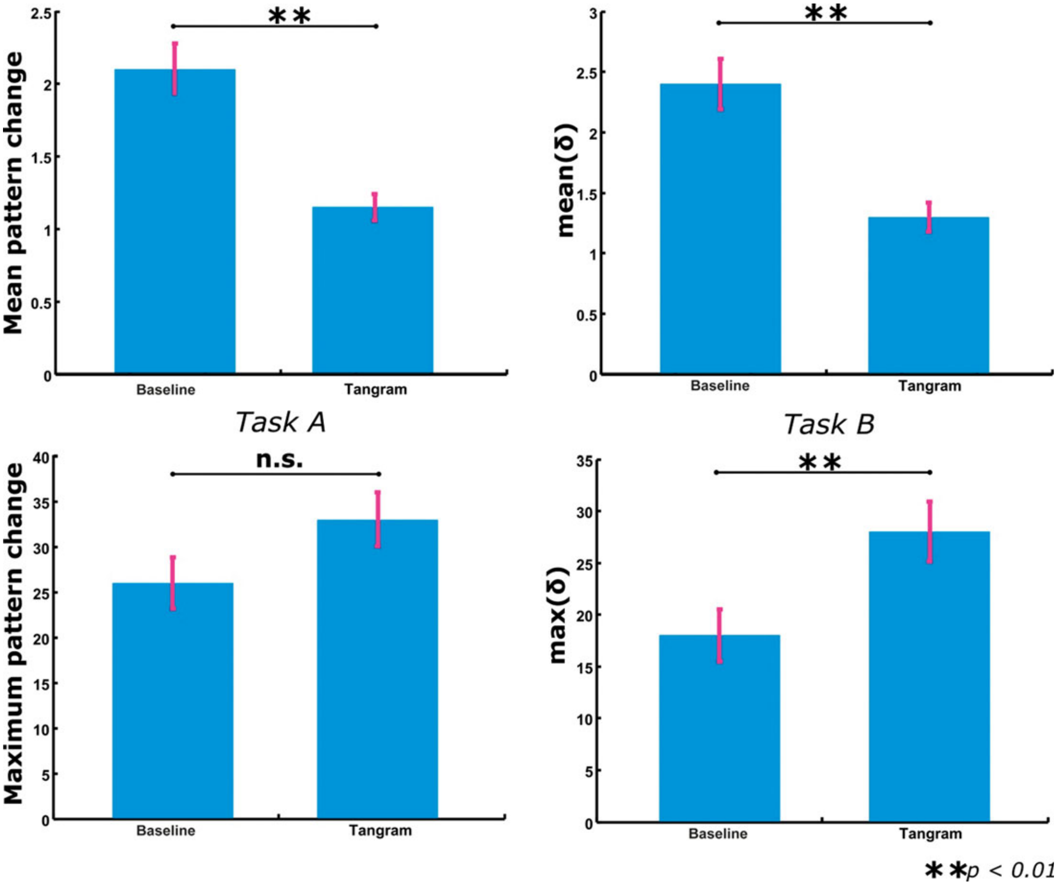

Fig. 6. Metrics for measuring spatial chunking behavior show statistically significant differences between the baseline and tangram group. Subjects in the tangram group showed smaller changes in gaze patterns (top panels) but the changes that did occur were more drastic in magnitude (bottom panels) across both area calculation tasks. Error bars represent ± 1 *SEM* within populations.

2.1.6. Discussion

The results from the analysis of task-oriented eye movements indicate that there are significant process differences between the two groups. Particularly, the tangram group appears to follow a recombining approach, partitioning the figure in a highly changing fashion, starting with components bigger than the given unit. The baseline group, on the other hand, appears to follow a less flexible counting process, starting with the given standard unit and smaller components.

This result only indicates that the use of the manipulative leads to a strategy change. It does not show that manipulation can lead to a better understanding of area. However, if the recombining strategy supports learning the key concepts involved in area, and also helps integrate these concepts, then manipulation could possibly improve the understanding of area.

Since our focus here is characterizing the changes in the cognitive process induced by the manipulative, rather than the specific ways in which the tangram manipulation task supports the complex integration required in learning of the area concept, our next study examined further the nature of the process change induced by the manipulative.

2.2. Study 2

The first study illustrated two points. One, the general direction of the change in cognitive process induced by the manipulative task is indicated at the macro-level by the qualitative analysis of hand and pencil movements and self-reports of strategies used. Two, this process change indicated by the macro-level analysis can be more clearly captured by a micro-level analysis of task-oriented eye movements, as the nature of the change (in actions and moves in the problem space) can be characterized in more detail using the eye movement analysis method we have developed. In this follow-up study, we sought to characterize the connection between the manipulation and the problem-solving process in more detail. Specifically, this study asked the question: *Which aspect of the manipulation task, actions or structure, is leading to the shift in the problem-solving process?*

In particular, we were interested in finding out whether manipulation alone could lead to the difference we observed in study 1, or whether the geometric structure of the tangram is also needed. If the geometric structure is needed, the manipulation task is working in a coagulative fashion, combining both structure and actions from the manipulation task, and mapping this coagulated structure to the area task. If just physical manipulation is enough, then the shift in process could come just from actions. Particularly, there is something about the process of executing actions that leads to a shift in the cognitive process. One possibility here is the integration/binding ability of actions (Kothiyal et al., 2014; Majumdar et al., 2014). Since actions require constant and real-time integration of both motor elements and sensory elements, actions have an in-built integration capability, which would be activated by the manipulation task. This activation could then transfer to the area task, leading to a shift in cognitive process, particularly to a strategy where integration is a key feature. Note that support for the former coagulative possibility, which is what we found, does not rule out the integrating role of actions.

2.2.1. Procedure

Ten new sixth-grade participants with similar age profiles as the previous study were recruited (all from Marathi medium schools). They did the area study with a new pre-task: manipulating clay dough into any figure they wanted. All other design elements were kept the same as the first study. The figures students chose to make were animals, birds, or flowers (see Fig. 7). There was no baseline group in this case, as the results from the first baseline group could be compared with the clay group.

If extended restructuring actions based on any manipulative are enough to shift the strategy in the direction seen in the first study, the clay manipulation would lead to a pattern of task-oriented eye movements similar to the tangram group. If the geometric

Fig. 7. Clay models made by students.

structure of the tangram task is one of the important elements that contribute to the shift in strategy and the associated eye movement pattern, the clay task would not lead to this shift in the problem-solving process. The results turned out to be more nuanced than these trends suggested by reasoning.

2.2.2. Results

Here we report results comparing data from both studies. The clay group's accuracy performance was similar to that of the other two groups. Given the way the task-oriented eye movement analysis supported the qualitative analysis in the earlier study, only the eye data were analyzed to understand the problem-solving process in this condition. Comparing the results from both studies, interesting and nuanced differences (see Fig. 8) emerged across the groups. Building the clay model shifted the average change in gaze pattern in much the same way as the tangram manipulation, such that mean(δ) is statistically indistinguishable between the clay and tangram conditions (Task A: $t_{19} = 1.26$, $p = .23$, Task B: $t_{19} = 1.15$, $p = .26$). However, in the tangram condition, this drop in average gaze transition frequency was accompanied by the occurrence of large transitions (across chunked sub-units of the diagrams). In the clay condition, the opposite pattern is seen—the size of the largest transition is quite significantly lower in this condition than the baseline case (Task A: $t_{19} = 2.48$, $p = .02$, Task B: $t_{19} = 1.5$, $p = .15$), and clearly lower than in the tangram case ($p < .001$, for both problem tasks).

These results show that building the clay model reduces the average transition occurrence to the same degree as the tangram manipulation, but it does not change the size of the largest transitions. The task-oriented eye movement patterns of these participants thus indicates lesser visual exploration of the diagrams. Overall, these results suggest that participants primed using clay-modeling did use chunking (as they solved the problem using fewer moves than the baseline group), but to a much lesser extent than the tangram group, and using a less global (whole figure) and changing process (as they did not chunk far away elements).

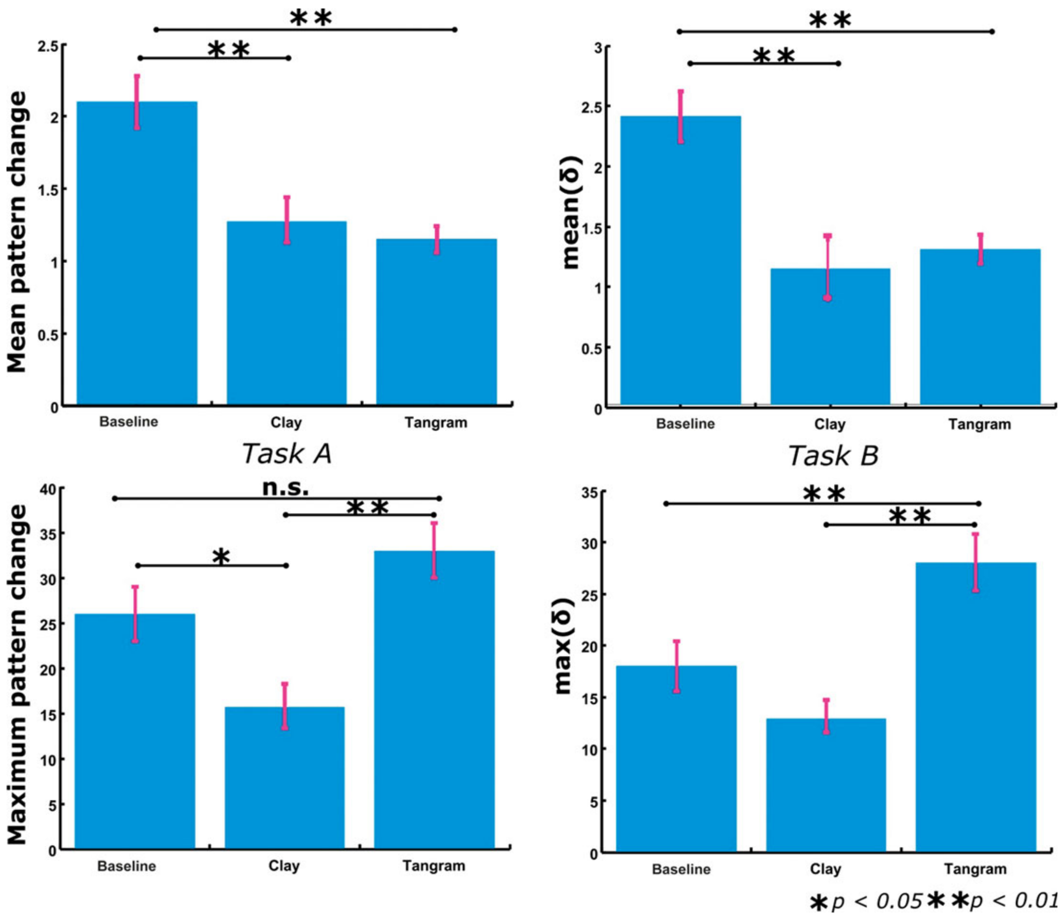

Fig. 8. Combined results from both studies. The clay condition leads to less transitions, similar to the Tangram condition (top panel). However, the movements are not as spread in space as the tangram condition (bottom panel).

Fig. 9 shows the results from the standard saccade analysis (results from both studies). The results are in the same direction as the task-oriented eye movement analysis above. As in the case of the standard analysis in the tangram case, there are no significant differences in this analysis as well.

One possible concern with these results—particularly when reflecting on these studies from the perspective of our finding that geometrical structure is needed for the Tangram effect—is that the tangram task has geometry, while this element is missing in the clay and knowledge tasks. This difference may have worked as a possible confound, as the effect we are reporting could derive from the presence/absence of geometry.

However, this "only geometry" interpretation requires concluding that action is not required for learning mathematical concepts such as area and fraction, and the actions on manipulatives are superfluous. Since the consensus in the literature is that actions are central

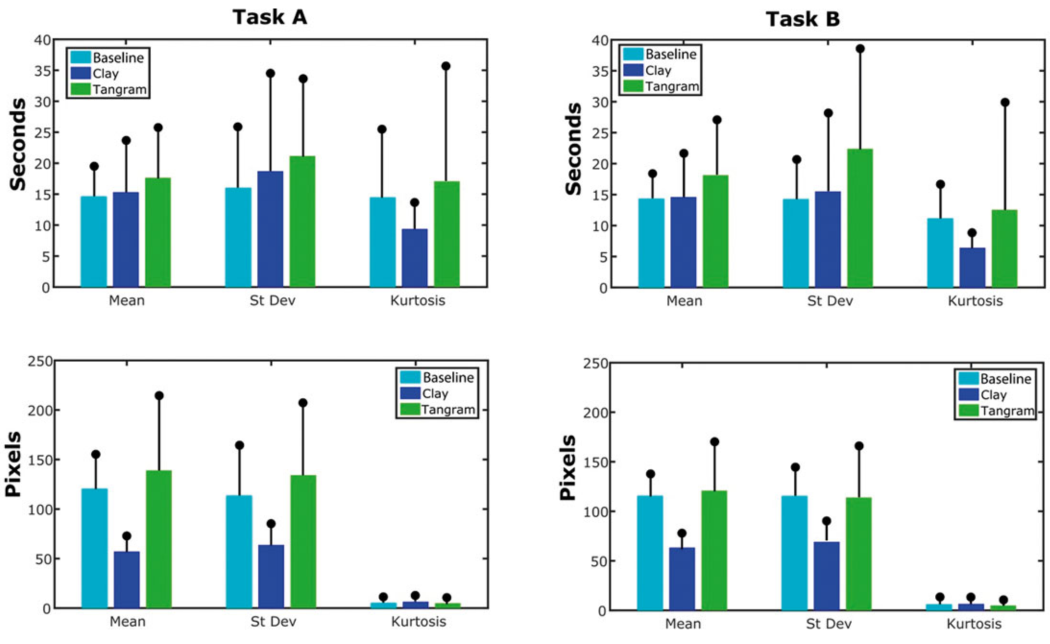

Fig. 9. Combined results from the two studies. Fixation duration and saccade length statistics for the baseline, tangram, and clay groups. Bars show standard deviations for the specific statistic. None of the differences reach statistical significance.

to the learning effects based on manipulatives, the geometry interpretation is not very persuasive.

Secondly, note that the "geometry-less" groups (clay and knowledge test) in study 1 and 2 show different eye movement patterns. This suggests manipulation does have an effect, as the clay study has manipulation, while the knowledge test does not. Since manipulation has an effect in the non-geometry clay case, it is likely that the effect in the Tangram case (the geometry case) also comes *partially* from the manipulation. Since this partial effect is all we are claiming, the possible confound based on geometry does not undermine our results.

2.3. Study discussion

The eye tracking analysis method we use is a simpler variation of the one used by Anderson (2012) to track the second-by-second thinking while students solved algebra problems. This type of process analysis is mostly done with neural response datasets (King & Dehaene, 2014). The method we present here extends this analysis to eye tracking data, particularly task-oriented eye movements during problem solving, where the focus is on the role of the eye as an actuator.

A significant chunk of eye tracking studies, particularly in user research, focus on where someone is looking in a given figure or text during task performance, and these data are usually used to understand the way information is distributed in the given figure or area, and how this distribution grabs visual attention. In such studies, the strong relationship between eye position and visual attention is used as a bijective map between eye movement and cognition. But this is not the only possible way to link eye movements to cognition. Eye tracking studies examining problem solving, as well as studies in education, also consider eye movements as driven by the requirements of the task (Epelboim & Suppes, 2001; Inglis & Alcock, 2012; Schneider, Maruyama, Dehaene, & Sigman, 2012; Smith, Mestre, & Ross, 2010; Susac, Bubic, Kaponja, Planinic, & Palmovic, 2014). The analysis we report here follows this approach, considering eye movements as task-oriented actions, changing in relation to the moves in the given task (similar to hand movements). The saccadic analysis we report for both studies, on the other hand, treats eye movements as context free, and as indicating shifts in just visual attention, which is considered as directed by information in the given figure. Our task-oriented analysis does not deny this role of attention, which focuses on the role of the eye as a perceptor. The task-oriented approach we take just shifts the focus to the role of the eye as an actuator, and the role played by executive attention in controlling the eye in this role. The task-oriented movements of the eye provide detailed process information about the way the task is performed (such as which point was visited, the sequence of visits, integration of visited points, etc.), thus capturing the most significant actions while solving the area problem.

In summary, using eye movement data to understand changes in the process of problem solving requires associating task-oriented eye movements with cognitive markers (such as strategies), which can be isolated using qualitative studies of problem solving. The characterization approach we report here combine such qualitative studies with eye tracking, and this could be a very productive way to gain insight into the micro-level changes involved in problem solving.

3. A theoretical model

The two studies revealed nuanced task-oriented eye movement patterns linked to strategy use across the three intervention conditions (baseline, tangram, clay), and the results suggest the following:

1. Systematic manipulation of any material before the area task can prime an action level shift in the problem-solving process, just through the actions involved in the manipulation (as seen in the lower transitions in both the clay and tangram case).
2. However, a more strategy-level shift is primed when there is structure embedded in the manipulation task. The embedded structure leads to a systematic pattern of manipulation actions, which, in turn, lead to systematic shifts in the problem-solving process (larger transitions across the figure seen in the tangram task).

In the following section, we develop a two-step model that accounts for both these results. Note that this model is presented as a general model of how manipulatives change problem-solving, even though the data used to derive it comes from a single representative study. Section 5.4 discusses the rationale for proposing such a general model, and the limitations of this proposal.

3.1. Recombinant enaction: How manipulatives generate strategy shifts

In the general cognitive model we propose below, the mechanism underlying manipulative-based learning, and learning-by-doing in general, is an augmentation of the "mutability" (Kahneman & Miller, 1986) or the "slippability" (Hoftstadter, 1985) of the problem-solving trajectory. This change in mutability is brought about by latent actions from the manipulation, which are carried over to imagination, particularly to executive attention operations in working memory. In this model, the actions during manipulation first primes the action system, and this process then expands the "action space," that is, the space of actions available while doing the problem task.

Next, when the problem task is encountered (in our case, the solving of the area problem), task-oriented eye movements (exploring the task space) and imagined actions (to restructure the task space) are generated. Note that this shift of actions to imagination occurs because the task elements in the given figure are *not manipulable* physically. During this exploration-and-imagination process, the primed actions (latent from the manipulation task) introduce *branch points* into the stream of imagined actions. This recombination process (recombining stored actions and imagined actions) creates new trajectories in the problem-solving space, which eventually leads to a problem solution.

This model suggests that the shift in the problem solving process does not come from a direct one-to-one mapping between the actions in the manipulative and the actions in the problem-solving process, as proposed by Hall (1998). Instead, the shift comes from a recombination process, where actions executed on the manipulative are combined with imagined actions. This recombination process extends the "action space" of imagined actions (see Chandrasekharan, 2009, 2014, for related discussions).

In this proposal, all latent actions carried over from the manipulation task will generate branch points in the imagination process. This accounts for the first result from the clay condition, which indicates that actions have an effect by themselves. However, because the long-distance movements seen in the tangram condition were not seen in the clay condition, *all* actions in the manipulation task (which translates to many generated branch points in the imagined action process) do not lead to the actual shift in the problem-solving process seen in the tangram case. This suggests the shift in the problem-solving process seen in the tangram case is dependent on an action-structure coagulation, combining the primed actions and the structure embedded in the actions. That is, only actions with certain embedded features (geometric features in the tangram case), particularly features that fit the action-possibilities provided by the task space (area in our case), lead to the problem-solving process shifting to a new pattern. The next section discusses neural mechanisms that could support this effect.

In sum, all manipulatives prime the action system, and these latent actions generate branch points in the imagined task space. Manipulatives thus work, in general, as systems that generate actions that could be recombined to generate new imagined procedures.

They thus expand the action space–the space of actions possible in the imagined task environment (Step 1 in Fig. 10). However, which branch points in the recombined action space are actually chosen for problem-solving depends on the saliency of the branch points. This saliency is driven by a matching, between the action possibilities provided by structures in the task space (affordances, Gibson, 1977, 1979) and the latent actions carried over from the manipulation task (Step 2 in Fig. 10). Thus, manipulatives that extend the action-space productively, that is, in such a way that the generated imagined actions match the action-possibilities provided by the task space, are more likely to generate interesting shifts in strategy. Given this mapping (between the task space structure and the structure embedded in action traces), the shifts in strategy generated by the manipulative may turn out to be moves in the task environment that could lead to possible solutions.

4. A mechanism account

Which neural mechanisms could support this model? We first outline three possible mechanisms, and then examine the way they map to the different stages of the proposed recombinant enaction model.

4.1. Common coding

Recent research in cognitive science and neuroscience shows that when humans perceive and imagine movements, particularly actions, the motor system is activated implicitly. In the other direction, perception and imagination of movements are improved by execution of movements. This three-way influence is explained by the common coding hypothesis, which proposes that the perception, execution, and imagination of movements share a common representation (common coding) in the brain.

First articulated clearly by Prinz (1992), this common neural representation allows any one of these movement representations to automatically trigger the other two movement representations (Prinz, 2005; Sebanz, Knoblich, & Prinz, 2005; also see Decety, 2002; Hommel, Müsseler, Aschersleben, & Prinz, 2001). One central outcome of common coding is a body-based "resonance"—the body instantly replicates all movements it detects, generating an internal representation that is dynamic, and based on body coordinates. This replication creates a dynamic trace, which can play a role in later cognition. All the replicated movements are not overtly executed. Most stays covert, either because the covert activation lacks strength, or because the overt activation is inhibited.

The basic argument for common coding is an adaptive one, where organisms are considered to be, fundamentally, action systems. In this view, the cerebrospinal system evolved in multi-cellular organisms to solve the problem of coordinating actions, which required the integration of a large number of cells (Llinas, 2001). Perceptual and cognitive systems evolved later, to plan and predict the outcome of such integrated movements. These higher-level planning and prediction systems are therefore dynamically coupled to

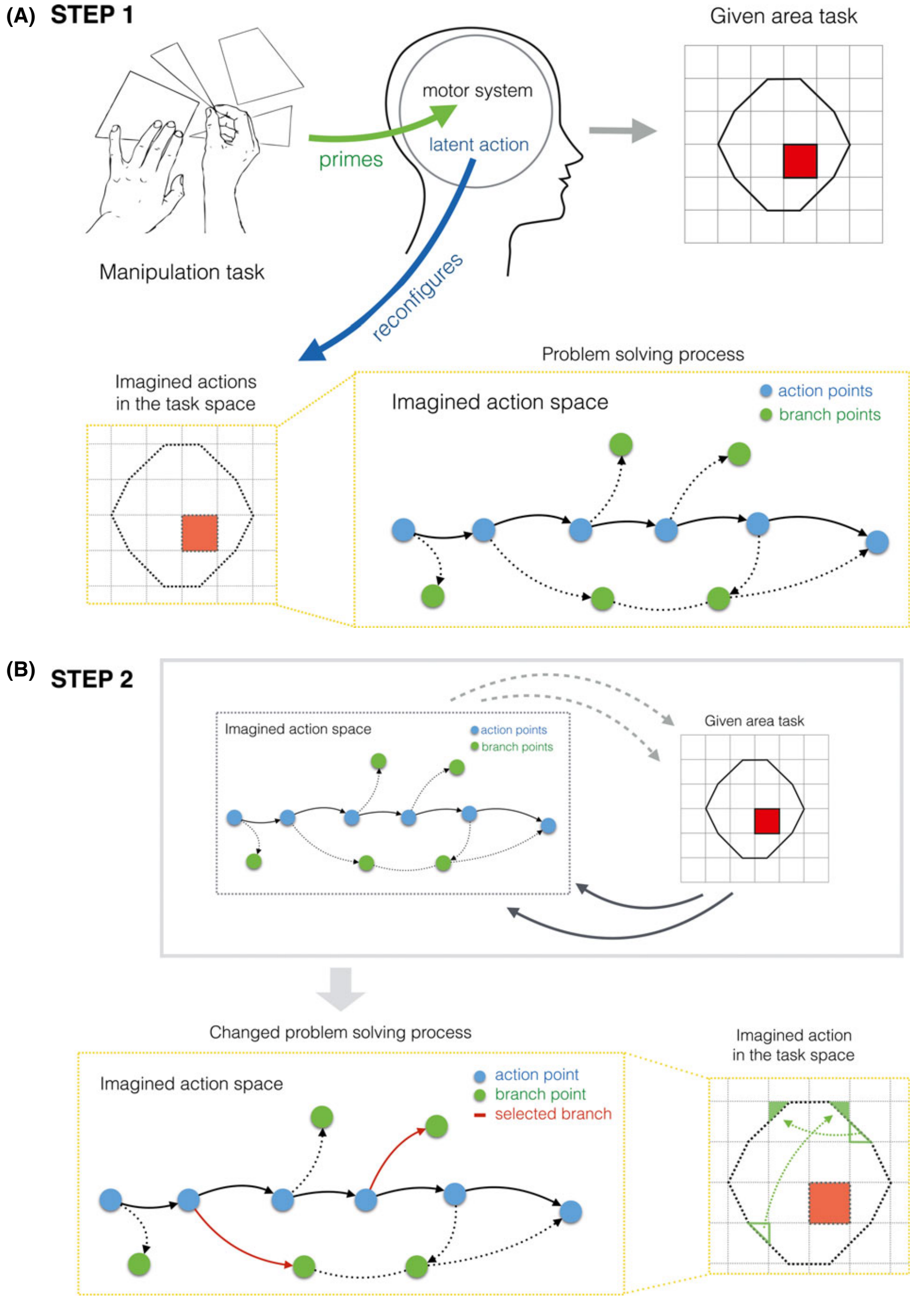

Fig. 10. The recombinant enaction model. Step 1 illustrates the way actions in the manipulative primes the motor system and the way these latent actions adds branch points to the imagined problem-solving process. Step 2 shows how interaction with the affordances of the area figure leads to some of the branch points getting selected, based on a matching between the affordances of the task space and the generated branch points, and the way this matching changes the problem-solving process.

action systems, in ways that help organisms act quickly and appropriately. Common coding, and the resultant replication of external movements in body coordinates, provides *one* such highly efficient coupling for planning and prediction. In this evolutionary view, common coding (of execution, perception, and imagination of movements) is to be expected, as evolutionary trajectories are influenced by existing systems, and there is a strong bias toward reusing existing systems for new functions, such as imagination of movement. The two mechanisms outlined below are ways in which existing common coding structures get extended, to support new functions.

4.2. Forward models

Building on the model of common coding, recent approaches to motor control propose that actions are controlled using "internal forward models (predictors) that emulate the dynamic behavior of our body and environment, thereby capturing the forward or causal relationship between our actions and their consequences" (Schubotz, 2007). The idea of forward models (Wolpert & Kawato, 1998) suggests that all actions involve a minimal imagination element (see dotted pathway in Fig. 11), which seeks to predict the consequences of the action. This minimal imagination system provides a seed process to develop a more full-fledged imagination of movements.

Interestingly, forward models could run independent of overt actions, and this "off-line" process could contribute to motor imagery, estimation of outcomes of different possible actions, and evaluation as well as development of motor plans (Grush, 2004). Forward models are also considered to have a role in the estimation of dynamic properties of manipulated objects (Davidson & Wolpert, 2005). Important for our purposes here, Schubotz (2007) proposes that the forward modeling system is used to predict movements of inanimate objects (such as waves landing on a beach) and serial events (such as sequential moves in a problem-solving task). This prediction process works by *automatically* activating the closest forward model to the movement that is perceived. This closest model, activated in a resonant fashion, is then tweaked based on input from the perceptual system, to make predictions about the encountered inanimate movement. The resonated forward models are then, in turn, revised based on feedback from this prediction process. Given common coding between execution, perception and imagination of movements, the above principles (resonant activation of closest forward model, prediction based on perceptual feedback, revision of forward model) would also apply to "off-line" imagination, of inanimate object movement and serial events, using forward models.

4.3. Extension of body schema through tool use

A number of studies in monkeys have shown that the body schema is extended to incorporate external objects, particularly tools (for a review, see Maravita & Iriki, 2004). One influential study (Iriki, Tanaka, & Iwamura, 1996) examined the firing of visuotactile bimodal neurons before and after a monkey learned to use a stick to gather food. Bimodal neurons in the intra-parietal cortex respond to both somatosensory and visual input on or

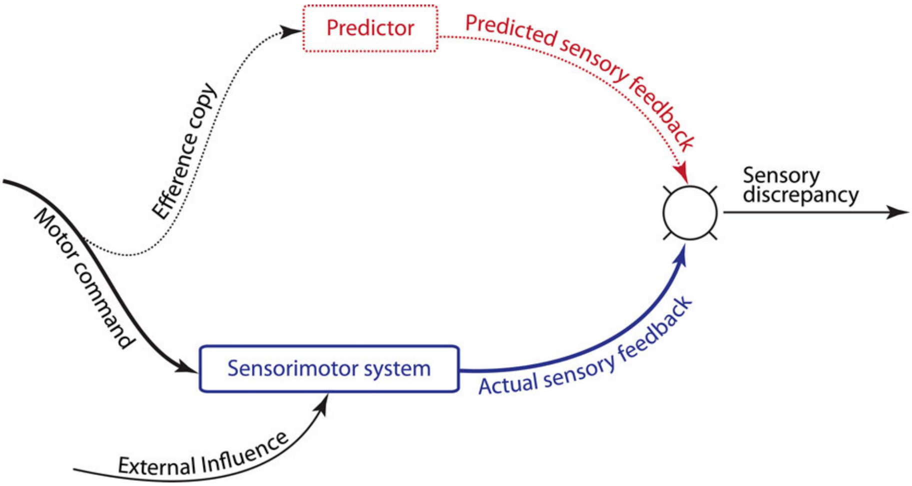

Fig. 11. The schema for a forward model. The thick line indicates the activation of the sensorimotor system, and the dotted line indicates the predictive model, which is based on a copy ("efference copy") of the sensorimotor activation. The discrepancy between the actual and predicted feedback is used to update both the forward model and the motor plan. Adapted from Schubotz (2007).

near the hand. That is, the bimodal neurons coding for the hand area will fire when the hand is touched, as well as when a light is flashed on the hand. Interestingly, this firing happens when the light is flashed not just on the hand itself, but also in the space close to the hand ("peri-personal space"), indicating that the neurons code for *the space of possible activity*, rather than just the hand. Iriki et al. (1996) examined whether this firing pattern changed when the monkey started using a stick as a tool. This investigation was done in three phases. In the first phase, there was no stick, and the light was flashed on and near the hand, and the bimodal neuron fired for both. In the second phase, the monkey passively held the stick, and the investigators flashed the light near the monkey's hand, as well as at the end of the stick. The bimodal neuron fired only when the light was flashed near the hand. In the third phase, the monkey used the stick to retrieve food from a location that was not reachable by its hand. Immediately after this action, the investigator flashed the light on the hand as well as at the end of the stick. The bimodal neuron now fired for light flashes near the hand as well as at the end of the stick (and in a later study, to in-between points on the stick), showing that the peri-personal space (the area of possible activity coded for by the neuron) has been extended to include the area covered by the stick. The action led to the stick being incorporated into the body, and the monkey's peri-personal space (possible activity space) now extended to the entire area, and objects, reachable by the stick.

This extension of peri-personal space shows that such incorporation is not just about adding an external entity to the body schema. Incorporation expands the range of *possible activities* the monkey can do—in terms of location of activity, other entities involved, nature of activity, the number of activities, and the permutations and combinations of activities. This expanded range also extends the monkey's forward models, thereby changing its understanding/knowledge of the possibilities of the stick, as well as the space around it, which is now understood in relation to the stick. The monkey's cognitive capacities are thereby expanded, particularly its capacity to plan new actions in the extended action space, using the revised forward models, which incorporate the stick (see Chandrasekharan, 2014, 2016, for a wider discussion). Similar incorporation of external entities into the body schema has been shown with humans as well (Farne, Iriki, & La'-davas, 2005).

4.4. A mechanism account of the recombinance model

How are the different steps in the recombinance model supported by the above mechanisms? There is no one-to-one mapping between the stages of the model and the above mechanisms. Rather, each stage requires support from all the mechanisms. The recombinance model has three steps: action priming, mutation of the problem-solving trajectory (to generate branch points), and selection of branch points (for actions involved in problem-solving). Broadly, the first two extend action spaces, and the third recombines action spaces. Below we examine how these steps are supported by the above mechanisms.

4.4.1. Action priming

In this process, traces of the actions done on the manipulative are stored, and these traces are activated in imagination during problem-solving. At the mechanism level, this priming process, where execution of an action primes imagination of another similar action (Chandrasekharan, Binsted, Ayres, Higgins, & Welsh, 2012), is supported by the common coding mechanism, as it connects execution and imagination of movements.

At a more operational level, forward models are required for the execution, trace and imagination steps. The forward model would be activated while planning and executing the actions with the manipulative. Following this activation, the trace of the manipulation, and its effects, would be stored in this forward model. Since forward models are activated for motor imagery, activation of the stored trace in imagination would be based on the offline activation of the forward model.

Moving to a higher level of description, the manipulative is similar to a tool (as it is used to reach a goal), and this suggests the actions with the manipulative would extend the body schema, and thus the action space, of the learner. This extension of the action space would also expand the action space of the forward model related to the manipulation, which leads to wider imagined action possibilities when this forward model is activated and run "offline" during the imagination process.

4.4.2. Mutation of the problem-solving trajectory

The possible mechanism processes here are complex. We outline them in two steps, even though they would happen simultaneously.

In Step 1, when the area problem is encountered, the stored forward model from the manipulation process (the trace) is automatically activated, for two reasons. One, the forward model encodes a trace of the action immediately preceding the imagination process, and this trace is still active when the area problem is encountered—it is thus primed. Second, the area figure is unmanipulable, so solving the problem requires imagining the process of restructuring the components. For this imagination process, the forward model of the manipulation is activated—because it provides the closest approximation for such restructuring. The forward model associated with the manipulation (with the expanded action space from actions with the manipulative) is thus recruited automatically when imagining problem-solving actions on the given unmanipulable figure.

In Step 2, the imagination space is extended, to include the actions from the manipulative, as the recruited forward model comes with an extended action space (from the bodily incorporation of the manipulative). This is equivalent to the problem-solving trajectory being mutated, to generate branch points, where *all* the actions carried over from the manipulation process are included.

4.4.3. Selection of branch points

In the recombinance model, all the new branch points generated in imagination are not selected during the problem-solving process, only the ones that contribute to problem-solving are selected. How does the system know which branch points would contribute to problem-solving? This selection is driven by feedback and resonance, where the action possibilities (affordances) provided by the encountered area problem matches, and thus automatically activates, some of the actions available in imagination. The encountered action-possibilities thus make salient some branch points in the imagination space, specifically the branch points that embed actions supported by the elements in the external task space. In other words, the structural features of the given area problem make salient certain action features in imagination, and these are action features that embed the encountered structure.

The possible mechanism supporting this selection process is a complex offline variant of the way forward models are tuned, through comparison with executed actions and their effects. In the offline variant of this tuning process, two forward models are co-activated, both running in imagination. One is the imagined actions needed for problem-solving. The second is imagined actions activated by the action possibilities of the problem-space. These two processes would activate the same forward model, given the similarity principle (Schubotz, 2007). And because of the shared forward model, the two processes would be coupled. This coupling would lead to the problem-solving forward model (running in imagination) being revised, based on input from the action-possibility forward model (also running in imagination). The problem-solving forward model is the one that is revised because it is the one that is goal-directed. Once this forward model is revised, the executed actions will include only the selected branch points. Note that this revision process is not explicit, or based on a recognition process—it would be an automatic pruning based on resonance between the activated actions.

The above mechanism account of the three phases of the recombinance model is based on general neural mechanisms. It provides a possible neural basis to the proposed recombinance model of how doing becomes thinking–that is, how actions in the manipulative move to the imagination and get recombined, thus changing the problem-solving process.

In the next section, we examine how this account offers new ways of approaching the three cognition issues outlined in the Introduction.

5. General discussion

As outlined in the Introduction, manipulative-based learning connects together three major areas of cognition (distribution of cognition, the transfer process, embodied cognition). We discuss below how the recombinance model expands our understanding of these processes. Since the proposed model provides a mechanism account, the implications of the model for each of these areas are examined from a mechanism point of view. In each case, we focus on new insights provided by the recombinance model.

5.1. Distribution of cognition

Cognitive load is often distributed during problem solving, through the generation of epistemic actions and epistemic structures (Alibali & Nathan, 2012; Chandrasekharan & Nersessian, 2015; Goldin-Meadow & Wagner, 2005; Goldin-Meadow et al., 2009; Hutchins, 1995; Kirsh, 2010; Kirsh & Maglio, 1994). The cognitive and neural mechanisms that support the generation of such actions and structures, and the way these generation processes are integrated with the problem-solving process, are not well understood. The recombinance model offers a possible mechanism account of this process.

The recombinance model outlines how common coding and forward models support the development of internal traces, which embed action elements from the manipulation. The idea of action-embedded internal traces has wide experimental support. For instance, imaging studies on practitioners of mental abacus (expert abacus users who can do operations on an imagined abacus to solve arithmetic problems) show that visual and motor areas are activated while solving arithmetic problems based on the internalized abacus (Chen et al., 2006; Hanakawa, Honda, Okada, Fukuyama, & Shibasaki, 2003; Hatano & Osawa, 1983). The internal trace of the manipulative improves users' performance during complex arithmetic operations, as the operations in the internally stored manipulative are visuomotor, and thus separate from working memory. Visuomotor operations on the internal trace of the abacus trigger complementary actions such as gestures, and these actions help in calculation, as indicated by the drop in performance when mental abacus is done under motor interference (Frank & Barner, 2012; Hatano, Miyake, & Binks, 1977).

Recent work in embodied approaches to language argue for similar action-embedded internal structures that trigger overt actions. Words and sentences embed movements and experiences, and this embedding allowing language (declarative knowledge) to both trigger movements as well as incorporate movements (Bergen & Wheeler, 2010; Bub & Masson, 2012; Glenberg & Kaschak, 2002; Matlock, 2004; Pulvermuller, 2001; Thomas & Lleras, 2007; Wilson & Gibbs, 2007; Yee et al., 2013). Studies of eye movements while processing language indicate that the movements embedded in words and sentences are reenacted by the oculomotor system (Stocker, Hartmann, Martarelli, & Mast, 2015). Similarly, moving the eye in patterns similar to a target solution in an insight problem leads to participants solving the problem more often (Thomas & Lleras, 2007). Our related work suggests that the eye acts as a physical micro-simulator (Chandrasekharan et al., 2015), with eye movements working as a simulation of upcoming/needed body movements, thus providing real-time data that help in planning, executing, and imagining actions.

We propose that similar embedded action elements could trigger the distribution of cognition through the generation of epistemic actions and structures. For instance, Tetris players learn to execute novel actions on screen to lower mental rotations (and lower associated cognitive load) and to improve scores (Kirsh & Maglio, 1994). The following enaction steps, based on action traces, could account for the case of epistemic actions generated during Tetris playing:

- Whenever we work on a problem requiring imagination of movement (such as mental rotation or mathematical procedures), stored traces, that is, forward models, embedding closely related or similar actions from previous experience, are activated (Schubotz, 2007). For Tetris, these would be forward models of hand rotation movements—stored from interactions with objects and tools (such as bottles and screwdrivers)—as the imagined movements are rotations.
- This covert activation of closely related action traces (such as hand movements in the case of mental rotation) break out of inhibition occasionally, leading to overt activation of actions (Chandrasekharan, Athreya, & Srinivasan, 2010), particularly when the cognitive load is high (as in the case of advanced levels of Tetris).
- The overt actions thus generated (gestures, complementary actions etc.) sometimes lead to manipulation of task elements (such as zoids), based on the action-possibilities of these elements. This is similar to the recombination of the stored traces and action-possibilities in the area task. These overt actions contribute to problem solving in some cases (as in the case of physical rotation of zoids in Tetris).
- Through implicit reinforcement, these actions (which improve performance) become standard branch points in the problem-solving process, and are thus incorporated into the action space, becoming epistemic actions. These epistemic actions, in turn, become part of the forward model for that problem-solving process, to be activated whenever other similar problems are encountered.

This proposal extends the recombinance model to epistemic actions, by showing how distribution of cognition could emerge from the recombination of action traces with action possibilities provided by the task. Our account thus offers a way forward in understanding the mechanisms underlying distribution of cognition, particularly the generation of new actions and structures that lower cognitive load. Given the close connections between motivation and action (Heckhausen & Heckhausen, 2008), this model could also account for the way artifacts, and actions based on them, support motivation (Chandrasekharan & Tovey, 2012; Dutta & Chandrasekharan, unpublished data).

5.2. The transfer process

Transfer of a learned concept or skill, to an unrelated problem or context, is one of the key objectives of learning. Much of the discussion on transfer in the education and cognition literature seeks to understand the factors that lead to optimal transfer. There is an active debate on transfer, including redefining transfer (Bransford & Schwartz, 1999; Schwartz, Bransford, & Sears, 2005), different dimensions of transfer (Barnett & Ceci, 2002), and blocking overzealous transfer (Schwartz, Chase, & Bransford, 2012). Active learning, motivation, and diverse methods have been identified as important variables that mediate transfer (Goldstone & Day, 2012). Some of the design principles for developing interventions that support transfer include progressive formalization (Van Reeuwijk, 2001) and concreteness fading (Braithwaite & Goldstone, 2013; Fyfe, McNeil, Son, & Goldstone, 2014; Goldstone & Son, 2005; see also Son & Goldstone, 2009).

Transfer is thus a very active research area in education and cognition, but the cognitive/neural mechanisms underlying transfer are not well understood. The recombinance model, particularly the way the model accounts for epistemic actions, offers a way forward to identify these mechanisms. However, given that the recombinance model is an account of the cognitive process, it can only provide a mechanism account of the cognitive process of transfer, not the structural aspects of transfer. Current approaches to transfer, particularly far transfer, consider analogy as the central cognitive component (Bernardo, 2001; Perkins & Salomon, 1992). A dominant model of analogy (Gentner, 1983; Gentner, Rattermann, & Forbus, 1993) argues for a core set of structures that are shared by everyday analogies and scientific analogies. However, this analogy model is not enough to account for all cases of far transfer, particularly scientific discoveries (Chandrasekharan, 2013).

Different from the analogy approach to transfer, we argue below that epistemic actions, which systematically restructure the given problem, is an important component of the transfer process. In the area problem, the use of manipulatives leads to the expansion of the student's action space, as procedures that are enacted using the manipulative (breaking and reassembling the tangram) are incorporated into the action schema. When the symbol-based area problems are encountered, these stored procedures are primed, and they move to the imagination, leading to restructuring procedures—specifically breaking and reassembling of the figure elements—executed in the imagination. This model provides a preliminary account of how an enacted procedure moves to a new domain (manipulative to symbolic figure)—a case of transfer. Generalising from this, if many such manipulative tasks, in combination with problems with unmanipulable symbolic elements, are used while solving problems across a range of mathematics domains, systematic restructuring of any given symbol-based problem in imagination could become part of mathematical problem-solving in general.

Systematic restructuring goes much beyond imagination operations, particularly while solving far transfer problems, where a given problem (say circadian rhythm) is eventually understood as an instance of a general concept (oscillation). In most such real-world transfer situations, which general concept is applicable is not immediately apparent, and the given problem needs to be systematically restructured in multiple ways before some connection to the general concept becomes available (Nersessian & Chandrasekharan, 2009). In the case of oscillation (a widely applied general concept that is used to model a range of phenomena, including predator–prey population shifts, electrical circuits, chemical reactions, markets, and mechanical/civil engineering structures), the key restructuring involves representing the target behavior using a graph that shows the behavior as shifting above and below a baseline. This external restructuring is needed because the core concept of oscillation (as taught using the simple pendulum model) is not directly applicable to such far cases, as the behavior of the pendulum and its equation are not applicable to most other situations (such as electrical circuits and chemical reactions). The transfer of the oscillation concept to other situations is thus critically mediated by the generation of a specific external graphical representation (see also Chandrasekharan, 2009; Chandrasekharan & Nersessian, 2015), which requires systematically restructuring the given problem.

This analysis suggests that systematic restructuring of the problem (such as breaking and re-arranging elements, building models, generation of graphs, geometrical structures, equations etc.), is a key process in far transfer. This restructuring process is an instance of epistemic action, and epistemic structures (such as graphs) generated by these actions help distribute cognition. In this view of transfer, interaction is the central component, both while learning and using the core concept. This is because learning involves the generation and storage of manipulations and symbols, leading to an action-embedded amalgam of structure and procedures (Sfard, 1991; Tall, Thomas, Davis, Gray, & Simpson, 1999). Solving the problem involves the activation of these stored procedures, which leads to restructuring of the given problem.

Restructuring is a powerful strategy for problem-solving, as it can overcome functional fixedness (German & Defeyter, 2000), and thus leads to solutions to most solvable problems. Restructuring is not taught explicitly as a strategy in most school classrooms, but this ability is assumed, and required, by most transfer problems posed in exams. This includes standardized tests, such as non-standard figures (L-shaped figures, curved figures etc.) that are used to test the understanding of area. Restructuring is used widely by experts (Cross & Cross, 1998; Kothiyal, Murthy, & Chandrasekharan, 2016; Wankat & Oreovicz, 2015), and restructuring based on perception and simulation is a key element of solving equations, and possibly symbolic operations in general (Banerjee & Subramaniam, 2012; Landy & Goldstone, 2009; Landy et al., 2014). When systematically used across many situations, manipulatives may help in developing general restructuring capabilities, which are applicable across many problems.

Current approaches to transfer focus on the structural aspects of the core concept that is stored, and a pattern recognition operation (analogy) that is based on this structure. In this approach, understanding and working with a core concept (such as oscillation), across many situations, leads to a complex and rich conceptual structure, such that any given scenario with similar structure can be (quickly) classified as an instance of this stored pattern. In this pattern recognition view, transfer involves recognizing a structural essence (in encountered situations with similar structure) and then applying this essence structure to solve the problem. It is assumed that the target situations are quickly recognized as possible applications of the core concept (see Nersessian & Chandrasekharan, 2009, for a discussion). This recognition of the pattern, and then mapping the structure faithfully, is considered a marker of good/right learning.

The problem-restructuring view we propose above argues for a more interactive process, particularly during far transfer. In this view, the oscillation concept is an action-oriented dynamic network of structures and procedures, which is constituted by the sensorimotor operations done on symbols during the learning of oscillation (Landy et al., 2014; Sfard, 2000). The transfer process starts when the traces of these sensorimotor operations are activated, in a resonant fashion. In the first step of this process, many stored traces are activated when the learner is searching for possible solutions to the given problem. In the second step, a few traces resonate a limited or partial "fit," after systematic problem-restructuring actions are done on the given task. In the final step, one stored trace resonates better to the restructured version of the problem, and this lock-in starts structure-mapping actions. This stage may require many further restructuring steps, both in the internal network and the external task, to get a full "fit." In engineering design problems, restructuring is often done simultaneously on three fronts—the learned structure-procedure amalgam, the target problem, and the target result (Kothiyal et al., 2016).

As restructuring actions are epistemic actions, the recombinance model of epistemic actions offers a way of developing a mechanism account of the restructuring approach to transfer, starting from the expansion of action space during the initial learning of the core concept. The recombinance model also offers a way of understanding how sensorimotor operations on representations *constitute* the core concept (Landy et al., 2014; Pande & Chandrasekharan, 2017; Sfard, 2000). Importantly, this procedural and mechanism approach to transfer does not deny the structure and pattern-matching view—it just shifts the focus to the *gradual* process of transfer, particularly the role of restructuring and other epistemic actions in establishing the mapping between learned sensorimotor procedures and the procedures afforded by the problem task.

5.3. Embodied cognition

The recombinance model incorporates many elements from the broad theoretical framework of embodied cognition, particularly dynamic systems theory (coupling and resonance between traces, internal structures emerging from dynamic interaction), ecological psychology (action possibilities, that is, affordances, as a way of tuning the forward model), and enactive cognition (common coding, forward models, incorporation). Further, our approach to understanding the control role played by the eye during imagination is complementary to the "attentional anchor" role played by eye movements during embodied learning of proportions, where the learner creates proportional relations between his hands to make a screen green, using a gesture-based interface (Abrahamson, Shayan, Bakker, & van der Schaaf, 2016). In this ecological-dynamics approach, the manipulation and problem-solving are not separate, and the eye movement works as a way to advance the embodied interaction, as well as the real-time learning that develops through this interaction (see also Kirsh, 2011). In the analysis we report here, the manipulation and the problem-solving are separate phases, and we examine the role of eye movements only during the symbol-based problem-solving process. Combining this approach with the attentional anchor approach, it would be possible to examine correlations in eye movements during the manipulation and the imagination phases, and examine whether such correlations could be a marker for manipulative-based learning. This would help understand how action, attention, imagination and eye movement are interconnected in tasks where the manipulation and problem-solving are separate phases.

There is thus close alignment between our analysis approach and the dynamic systems approach. However, in a dominant interpretation of dynamic systems, this alignment is broken by the way the recombinance model builds on representational constructs such as stored traces, imagination, common coding, forward models and the extension of the body schema through tool use. These constructs, when considered as representational/symbolic, would put the recombinance model at odds with embodied cognition accounts that deny such internal representations (Chemero, 2000). Our approach would also be at odds with learning approaches inspired by this philosophical approach to dynamic systems (Abrahamson & Sánchez-García, 2016).

In the view that denies internal representations, interaction (with the external world or external symbolic structures) is both necessary *and sufficient* to explain cognitive effects. We accept the view that interaction is necessary, but not that it is sufficient, particularly while analyzing problems in education. We take this nuanced approach because the only theoretical constructs available in the sufficiency view are situated and embodied interactions, and these constructs, in our view, are not enough to explain the learning and use of concepts, as well as the role of imagination in both these processes (see Pande & Chandrasekharan, 2017). Also, it is unclear whether external representations are valid constructs in the radical accounts that reject representations. Pushed to the extreme, the radical view would require recreating Newton's situated experiences to learn Newton's Laws. Bruner argues that learning based on external representations, particularly the "telling" mode of learning, allows us to sidestep the extended experiential learning based on "showing" (Bruner, 1965). Representations thus appear essential for the project of education, which seeks to develop knowledge based on "compressed" experiences, and build on them. The radical rejection of representation would require denying the possibility of such compressed and representation-driven learning. Since the currently dominant form of education is driven by the "telling" mode, based on representations, and this method does produce learning effects, anti-representationalism is not a productive approach to understand this form of education.

From a more theoretical perspective, we consider the radical rejection of internal representations unnecessary, as a commitment to dynamics only requires a dynamic systems account of how internal representations emerge from activity. Internal structures can be compatible with dynamic systems if: (1) they are also dynamic systems; (2) they emerge from dynamic interaction with the world, and (3) they are coupled to dynamic interactions with the world, and constantly change with such interactions.

Such an account of internal representation is provided by Chandrasekharan and Stewart (2007). In this model, internal representations emerge dynamically from interactions with the environment, and they function as control systems similar to the Watt Governor (Van Gelder, 1998). They are neural networks (which are dynamic systems; Smolensky, 1988) that embed patterns emerging from constant interaction with the environment. These patterns orient the agent's actions in the environment, and the patterns, in turn, constantly change with the agent's actions. Further, the internal representations that emerge out of environmental interactions embed situations, dynamics and embodiment, and these stored traces would be available for deployment during later interactions with the world. This leads to a constantly learning embodied system.

The "offline" activation of these networks would involve re-enaction (or simulation) of the process of interaction that led to the embedded pattern (Chandrasekharan & Stewart, 2007). There is a range of converging evidence indicating that concepts and memory have this simulative nature (Bergen & Wheeler, 2010; Bub & Masson, 2012; Matlock, 2004; Pulvermuller, 2001; Wilson & Gibbs, 2007; Yee et al., 2013). In this dynamic view of internal representation, the generation of such representational structures is a state change (like from water to ice) that emerges out of dynamic interactions. This new "condensed interaction" state supports operations (such as imagination) that are not supported by the previous state. This dynamics-based reconceptualization of internal representations offers an account of internal traces, memory, and imagination that is compatible with dynamic systems views, as state changes are common in dynamic systems.

This way of understanding internal traces is compatible with the ecological psychology perspective as well, particularly if the traces have a common coding structure, as perception and action are co-constituted in common coding, with no representational processing *needed* in between (Chandrasekharan & Osbeck, 2010; Hommel et al., 2001). In this view, internal representations become control structures that "orient" attention (Vygotsky, 1980). Learning would then be the process that leads to the development of such control structures that orient attention, which directs actions toward specific aspects of the environment. In this control approach to knowledge, external representations, and manipulations on them, would be ways of tuning attention. This control view of knowledge supports educational approaches that emphasize the training of attention (Mason, 2003, 2008; also see Braithwaite & Goldstone, 2015; Goldstone, Landy, & Son, 2010) and self-control (Mischel, Shoda, & Rodriguez, 1989).

5.4. Limitations and future work

We present a new account of how doing becomes thinking, bringing together: (1) education and cognition questions, (2) a novel task-oriented combination of eye tracking, neural data analysis methodology, and qualitative methods (from education and problem solving), and (3) recent mechanism models from cognitive and motor neuroscience. The empirical and theoretical approaches we illustrate in this paper offers a starting point to understand the very complex process by which formal knowledge emerges from procedural knowledge, and the way these two knowledge systems interact in learning and discovery situations, particularly through the generation of epistemic actions and structures (Chandrasekharan & Nersessian, 2015). The model we propose offers a very preliminary account of the way manipulatives change the problem-solving process, and it offers a way to further develop cognitive and motor neuroscience approaches to problems in education (Howard-Jones, 2014; Varma, McCandliss, & Schwartz, 2008). In the following discussion, we outline some of the major limitations of the model and indicate some ongoing and future work to address these issues.

5.4.1. Grounding

Our mechanism account is constrained and grounded by data from a single illustrative study. However, given the role of manipulatives in connecting different areas of cognition such as distributed and embodied cognition and transfer, and their current application significance, we have proposed not just an account of our results, but a general account of how manipulatives change the problem-solving process. Since this model generalizes from the results of a specific study, it is very likely that the model would not account for other cases, such as situations where there are no separate manipulation and problem-solving phases, as in the case of Abrahamson et al. (2016). We partly address this generalizability issue by grounding the model in general purpose neural mechanisms (common coding, forward models, and extension of the body schema through tool use). Since these are general purpose mechanisms that connect action to imagination, they could account for data from other studies as well. However, in accounts of other data, the specific way in which these mechanisms come together would differ from the model we present here. Given this possibility, our general model is only illustrative, seeking to meet two pragmatic objectives:

1. Function as a seed model, working either as a convergence case (when other models are in line with the one presented here), or as a contrast case (when other models differ drastically from the one presented here).
2. Help develop theoretical guidelines for designing embodied media for learning mathematics and science (Abrahamson & Sánchez-García, 2016).

The model is thus just a starting point, and it will be revised significantly, or rejected, based on future studies.

5.4.2. Data

The model emerged from a focus on examining process data, which is very difficult to gather and analyze. This has limited the scope of the study we report, which is too small to make definitive claims about the validity of the model. Going forward, what kind of data could provide ways to validate or reject the model? The critical component of the model is the way the manipulative extends the action space of the learner. One way to validate or reject the model would be to examine whether working with the manipulative actually leads to the extension of the body schema, as in the tool use case. Recent work in cognitive neuroscience provides some experimental paradigms to examine this hypothesis (see Chandrasekharan, 2014 for a review), and we are currently developing studies based on these empirical approaches.

5.4.3. Domain

Manipulatives are used in many contexts other than mathematics, such as learning chemical structures, anatomy, engineering design, and so on. The role they play in such contexts would be quite different from the mathematics case, particularly because the operations in imagination are more concrete in these areas. Similarly, the role played by manipulatives in laboratories, and the way they change imagination in this context, would be very different from the above cases. The model we propose is thus limited to cases where manipulatives embed abstract procedures, and the way they change the process of solving procedural problems. More sophisticated models would be needed to account for cases of building (such as in bio-engineering), where material structures such as reaction pathways and tissues are built and manipulated.

Addressing these three limitations would require revising the proposed model significantly. However, we hope the current version provides a starting point to better understand the two-way interaction between actions and formal conceptual knowledge, and the way manipulatives mediate this interaction. A great deal of interesting work lies ahead.

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